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COMPACTNESS & THE TYCHONOFF THEOREM

Tychonoff's Theorem:

An arbitrary product $X = \prod_{\alpha \in \Lambda} X_\alpha$ of compact spaces X_α is compact.

We will prove this theorem using the lemma(s) below. But first:

Def: Let Y be a topological space. A collection $\mathcal{A} = \{A_i\}_{i \in I}$ of subsets of Y is said to have the finite intersection property (FIP) if for every finite collection $\{A_{i_1}, \ldots, A_{i_n}\} \subseteq \mathcal{A}$, $\bigcap_{k=1}^{n} A_{i_k} \neq \phi$.

Prop: A space Y is compact $\Leftrightarrow$ every collection $\mathcal{A}$ of closed sets having FIP satisfies $\bigcap_{A \in \mathcal{A}} A \neq \phi$.

Proof: $\Rightarrow$: Let $\mathcal{A}$ be a collection of closed sets with FIP.

Suppose $\bigcap_{A \in \mathcal{A}} A = \phi$

then $\bigcup_{A \in \mathcal{A}} (Y \setminus A) = Y$

$\{Y \setminus A : A \in \mathcal{A}\}$ is an open cover of Y.

$\because$ Y is compact, $\exists$ a finite sub-cover $\{Y \setminus A_1, \ldots, Y \setminus A_n\}$ of Y

$\Rightarrow \bigcup_{i=1}^{n} (Y \setminus A_i) = Y$

$\Rightarrow \bigcap_{i=1}^{n} A_i = \phi$ $\Rightarrow\Leftarrow$ fact that $\mathcal{A}$ has FIP

$\Leftarrow$: Suppose Y is not compact, it has an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ with no finite subcover. ie, $\forall$ $U_{\alpha_1}, \ldots, U_{\alpha_n}$ in $\mathcal{U}$, $\bigcup_{i=1}^{n} U_{\alpha_i} \neq Y$.

$\Rightarrow \mathcal{A} = \{Y \setminus U_\alpha\}_{\alpha \in \Lambda}$ is a collection of closed sets s.t. $\forall$ $Y \setminus U_{\alpha_1}, \ldots, Y \setminus U_{\alpha_n}$, $\bigcap_{i=1}^{n} Y \setminus U_{\alpha_i} \neq \phi$. $\Rightarrow$ $\mathcal{A}$ has FIP.

$\Rightarrow \bigcap_{\alpha \in \Lambda} Y \setminus U_\alpha \neq \phi \Rightarrow \bigcup_{\alpha \in \Lambda} U_\alpha \neq Y \Rightarrow\Leftarrow$.

Prop: Let Y be a set. Let $\mathcal{A}$ be a collection of subsets of Y with FIP. Then there exists a collection $\mathcal{D}$ of subsets of Y s.t.

1. $\mathcal{D}$ satisfies FIP
2. $\mathcal{D} \supseteq \mathcal{A}$
3. For every $\mathcal{D} \subsetneq \mathcal{D}' \subseteq 2^Y$, $\mathcal{D}'$ does not have FIP.

(ie, $\mathcal{D}$ is "maximal" w.r.t. FIP).

Proof:

The proof is going to use Zorn's lemma.

Stated as:

> **Zorn's Lemma:** Let A be a set that is strictly partially ordered (by $<$) in which every simply ordered set has an upper bound in A. Then A has a maximal element x. That is, there is no $y \in A$ s.t. $x < y$.
>
> Remark: Zorn's Lemma is equivalent to the Axiom of Choice.

Proof of prop continued:

Given $\mathcal{A}$ is a collection of subsets of Y with FIP,

$\mathbb{A} = \{ \mathcal{B} \supseteq \mathcal{A} : \mathcal{B} \subseteq 2^Y \text{ with FIP} \}$.

$\mathcal{A} \in \mathbb{A}$. So $\mathbb{A} \neq \phi$.

$\subsetneq$ imposes a strict partial order on $\mathbb{A}$.

Let $\mathbb{B}$ be a simply ordered subset of $\mathbb{A}$.

We want to show: $\mathbb{B}$ has an upper bound.

Let $\mathcal{C} = \bigcup_{\mathcal{B} \in \mathbb{B}} \mathcal{B}$.

Note:
1. $\mathcal{C} \subseteq 2^Y$ $\because$ $\mathcal{B} \subseteq 2^Y$ $\forall \mathcal{B} \in \mathbb{B}$.
2. $\mathcal{C} \supseteq \mathcal{A}$ $\because$ $\forall \mathcal{B} \in \mathbb{B}$, $\mathcal{B} \supseteq \mathcal{A}$.
3. $\mathcal{C}$ has FIP: Choose $C_1, C_2, \ldots, C_n \in \mathcal{C}$ ie, $C_1, C_2, \ldots, C_n \subseteq Y$. Need: $\bigcap_{i=1}^{n} C_i \neq \phi$

   For each i, $\exists$ $\mathcal{B}_i \in \mathbb{B}$ s.t. $C_i \in \mathcal{B}_i$.

   $\because$ $\mathbb{B}$ is simply ordered, WLOG, $\mathcal{B} = \max \mathcal{B}_i$.

   $\Rightarrow \mathcal{B} \supsetneq \mathcal{B}_i$ $\forall i$.

   Note: $C_1, \ldots, C_n \in \mathcal{B}$, which has FIP.

   $\Rightarrow \bigcap_{i=1}^{n} C_i \neq \phi$.

$\Rightarrow$ $\mathbb{B}$ has an upper bound.

So we can apply Zorn's lemma to $\mathbb{A}$. This gives us a maximal element $\mathcal{D}$ of $\mathbb{A}$.

This $\mathcal{D}$ has FIP, and $\mathcal{D} \supseteq \mathcal{A}$.

Lemma: Let Y be a set, let $\mathcal{D}$ be a collection of subsets of Y that is maximal w.r.t. FIP.

Then

1. Any finite intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$
2. If $A \subseteq Y$ is a subset that intersects every $D \in \mathcal{D}$, then $A \in \mathcal{D}$.

Proof: 1. Given $D_1, \ldots, D_n \in \mathcal{D}$, WTS: $D = \bigcap_{i=1}^{n} D_i \in \mathcal{D}$.

Suppose $D \notin \mathcal{D}$.

Take $\mathcal{D}' = \mathcal{D} \cup \{D\}$. This is a collection with FIP and $\mathcal{D} \subsetneq \mathcal{D}'$.

$\Rightarrow\Leftarrow$ fact that $\mathcal{D}$ is maximal.

2. $A \subseteq Y$ is s.t. $A \cap D \neq \phi$ $\forall D \in \mathcal{D}$.

If $A \notin \mathcal{D}$, take $\mathcal{D}' = \mathcal{D} \cup \{A\}$.

$\mathcal{D}'$ satisfies FIP and $\mathcal{D} \subsetneq \mathcal{D}'$.

$\Rightarrow\Leftarrow$ fact that $\mathcal{D}$ is maximal.

Proof of Tychonoff's Theorem:

$$X = \prod_{\alpha \in \Lambda} X_\alpha$$

Let $\mathcal{A}$ be a collection of closed subsets of X with FIP. We WTS $\bigcap_{A \in \mathcal{A}} A \neq \phi$.

$\exists \mathcal{D} \subseteq 2^X$ that is maximal w.r.t. FIP and $\mathcal{D} \supseteq \mathcal{A}$. It suffices to show

$$\bigcap_{D \in \mathcal{D}} cl(D) \neq \phi.$$

Given $\alpha \in \Lambda$, let $\pi_\alpha : X \to X_\alpha$ be the standard projection.

Consider the collection $\{\pi_\alpha(D) : D \in \mathcal{D}\}$.

This collection, and therefore, $\{ cl(\pi_\alpha(D)) : D \in \mathcal{D} \}$ has FIP in X_α

$\because$ X_α is compact, $\bigcap_{D \in \mathcal{D}} cl(\pi_\alpha(D)) \neq \phi$.

Pick $x_\alpha \in \bigcap_{D \in \mathcal{D}} cl(\pi_\alpha(D))$.

Let $\underset{X}{x \in} = (x_\alpha)_{\alpha \in \Lambda}$. We will show $x \in \bigcap_{D \in \mathcal{D}} cl(D)$

Let $x_\alpha \in U_\alpha \subseteq X_\alpha$ be open.

Then $x \in \pi_\alpha^{-1}(U_\alpha)$ is a sub-basis element for the product topology on X.

Then $\pi_\alpha^{-1}(U_\alpha)$ intersects every element of $\mathcal{D}$.

$\Rightarrow$ U_α intersects $\pi_\alpha(D)$ $\forall D \in \mathcal{D}$.

$\Rightarrow \exists$ $\pi_\alpha(y) \in U_\alpha \cap \pi_\alpha(D)$ for some $y \in \pi_\alpha^{-1}(U_\alpha) \cap D$.

$\because$ $\mathcal{D}$ is maximal, and $A = \pi_\alpha^{-1}(U_\alpha)$ intersects every element of $\mathcal{D}$,

it follows that $A \in \mathcal{D}$.

$\pi_\alpha^{-1}(U_\alpha) \in \mathcal{D}$.

Since $\mathcal{D}$ has FIP,

$\Rightarrow$ every sub-basis element containing $x = (x_\alpha)_{\alpha \in \Lambda}$ belongs to $\mathcal{D}$

$\Rightarrow$ every basis element containing x belongs to $\mathcal{D}$.

$\Rightarrow x \in \bigcap_{D \in \mathcal{D}} cl(D)$

Exercise: Find a collection of compact sets whose product in the box topology is not compact.